ON k-NONCROSSING PARTITIONS

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Abstract. In this paper we prove a duality between k-noncrossing partitions over $[n]=\{1,\dots,n\}$ and k-noncrossing braids over $[n-1]$. This duality is derived directly via (generalized) vacillating tableaux which are in correspondence to tangled-diagrams [6]. We give a combinatorial interpretation of the bijection in terms of the contraction of arcs of tangled-diagrams. Furthermore it induces by restriction a bijection between k-noncrossing, 2-regular partitions over $[n]$ and k-noncrossing braids without isolated points over $[n-1]$. Since braids without isolated points correspond to enhanced partitions this allows, using the results of [1], to enumerate 2-regular, 3-noncrossing partitions.

1. Introduction and Background

In this paper we prove a duality between k-noncrossing partitions and braids, a particular type of tangled-diagrams [6]. The duality implies a bijection between 2-regular, k-noncrossing partitions and k-noncrossing braids without isolated points, which are in bijection to enhanced partitions. We then compute the number of 3-noncrossing, 2-regular partitions over $[n]=\{1,\dots,n\}$, i.e. k-noncrossing partitions without arcs of the form $(i,i+1)$. The enumeration of 3-noncrossing, 2-regular partitions is not entirely trivial. This is due to the fact that the lack of 1-arcs translates into an asymmetry induced by the nonexistence of the pair of steps $((\varnothing,+\square_1),(-\square_1,\varnothing))$, where "$\pm\square_i$" denotes the adding/removing of a square in the ith row of the shape. We derive the above duality directly via the (generalized) vacillating tableaux [6] and prove its combinatorial interpretation in terms of the contraction of arcs, originally introduced by Chen *et.al.* in [5] in the context of a reduction algorithm for noncrossing partitions.

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Our results imply novel connections between different combinatorial objects and are of conceptual interest. For instance, Bousquet-Mélou and Xin [1] have enumerated 3-noncrossing partitions and 3-noncrossing enhanced partitions separately, using kernel methods in nontrivial calculations. By construction enhanced partitions correspond to hesitating tableaux [4] which accordingly enumerate braids without isolated points. Our duality theorem implies therefore that either one of these computations would imply the other. Furthermore our results integrate the concepts of vacillating and hesitating tableaux due to Chen *et.al.* [4]. 2-regular partitions are of particular importance in the context of enumerating RNA tertiary structures with base triples [9].

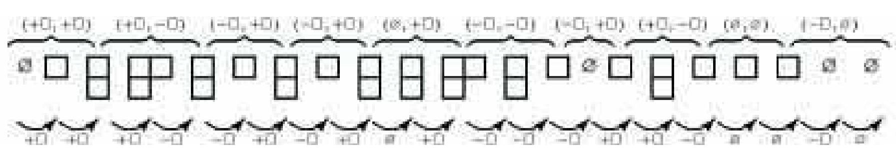
2. Tangled-diagrams and vacillating tableaux

In this Section we provide some basics on tangled-diagrams [6]. A tangled-diagram is a labeled graph, G_n, over $[n]$ with degree ≤ 2, represented by drawing its vertices in a horizontal line and its arcs (i, j) in the upper halfplane having the following properties: two arcs (i_1, j_1) and (i_2, j_2) such that $i_1 < i_2 < j_1 < j_2$ are crossing and if $i_1 < i_2 < j_2 < j_1$ they are nesting. Two arcs (i, j_1) and (i, j_2) (common lefthand endpoint) and $j_1 < j_2$ can be drawn in two ways: either draw (i, j_1) strictly below (i, j_2) in which case (i, j_1) and (i, j_2) are nesting (at i) or draw (i, j_1) starting above i and intersecting (i, j_2) once, in which case (i, j_1) and (i, j_2) are crossing (at i):

and of two arcs $(i, j), (i, j)$, i.e. where i and j are both: right- and lefthand endpoints are completely analogous. Suppose $i < j < h$ and that we are given two arcs (i, j) and (j, h). Then we can draw them intersecting once or not. In the former case (i, j) and (j, h) are called crossing, in the latter noncrossing arcs:

A k-noncrossing braid is a k-noncrossing tangled-diagram in which all vertices j of degree two are either incident to loops (j, j) or crossing arcs (i, j) and (j, h), where $i < j < h$. We denote the set of k-noncrossing braids over $[n]$ by $\mathcal{B}_k(n)$. For instance

A shape is a collection of squares, "$\Box$", arranged in left-justified rows with weakly decreasing number of squares in each row. A vacillating tableaux V_λ^{2n} of shape λ and length $2n$ is a sequence $(\lambda^0, \lambda^1, \dots, \lambda^{2n})$ of shapes such that (i) $\lambda^0 = \varnothing$ and $\lambda^{2n} = \lambda$, and (ii) $(\lambda^{2i-1}, \lambda^{2i})$ is derived from λ^{2i-2}, for $1 \leq i \leq n$ by either $(\varnothing, \varnothing)$: do nothing twice; $(-\Box, \varnothing)$: first remove a square then do nothing; $(\varnothing, +\Box)$: first do nothing then add a square; $(\pm\Box, \pm\Box)$: add/remove a square at the odd and even steps, respectively. Let $\mathcal{V}_\lambda^{2n}$ denote the set of vacillating tableaux, for instance,



We have the following bijection between tangled-diagrams and generalized vacillating tableaux [6] which integrates the notions of vacillating and hesitating tableaux of Chen *et.al.* [4]. In the following we refer to generalized vacillating tableaux simply as vacillating tableaux.

Theorem 1. *There exists a bijection between the set of vacillating tableaux of shape $\varnothing$ and length $2n$, $\mathcal{V}_\varnothing^{2n}$ and the set of tangled-diagrams over n vertices, $\mathcal{G}_n$*

$$\beta\colon \mathcal{V}_\varnothing^{2n} \longrightarrow \mathcal{G}_n \ . \tag{2.1}$$

Furthermore, a tangled-diagram G_n is k-noncrossing if and only if all shapes λ^i in its corresponding vacillating tableau have less than k rows, i.e. $\beta\colon \mathcal{V}_\varnothing^{2n} \longrightarrow \mathcal{G}_n$ maps vacillating tableaux having less than k rows into k-noncrossing tangled-diagrams. Furthermore there exists a bijection between the set of k-noncrossing and k-nonnesting tangled-diagrams.

Restricting the set of generating step-pairs of vacillating tableaux recovers the bijections of Chen *et.al.* [4]: Let $M = \{(\varnothing, \varnothing), (-\Box, \varnothing), (\varnothing, +\Box)\}$, $\mathcal{V}_{P,k,\varnothing}^{2n}$ and $\mathcal{V}_{B,k,\varnothing}^{2n}$ denote the set of tableaux with

less than k rows and generated by $P = M\dot{\cup}\{(-\square, +\square)\}$ and $B = M\dot{\cup}\{(+\square, -\square)\}$. Theorem 1 allows us to identify $\mathcal{V}^{2n}_{P,k,\varnothing}$ with $\mathcal{P}_k(n)$ and $\mathcal{V}^{2n}_{B,k,\varnothing}$ with $\mathcal{B}_k(n)$. For partitions and braids we have the following correspondences between the elementary pair-steps and associated tangled-diagram arc-configurations:

3. Main results

We now prove the duality between partitions over $[n]$ and braids over $[n-1]$. *A posteriori* the above bijection can be proved directly. However, we arrived at this interpretation studying vacillating tableaux of k-noncrossing partitions and braids.

Theorem 2. *Let $k \in \mathbb{N}$, $k \geq 3$. Then we have the bijection*

$$\vartheta\colon \mathcal{P}_k(n) \longrightarrow \mathcal{B}_k(n-1)\ , \tag{3.1}$$

where ϑ has the following property: for any $\pi \in \mathcal{P}_k(n)$ holds: (i,j) is an arc of π if and only if $(i, j-1)$ is an arc in $\vartheta(\pi)$.

Proof. A k-noncrossing partition π corresponds via Theorem 1 uniquely to a vacillating tableaux, $V^{2n}_{\varnothing}(\pi) = (\lambda^i)_{i=0}^{2n}$. Let $\pm\square_h$ denote the adding or subtracting of the rightmost square "$\square$" in the hth row in a given shape λ and let "$\varnothing$" denote doing nothing. $(\lambda^i)_{i=0}^{2n}$ uniquely corresponds to a sequence of pairs $\sigma_\pi = ((x_i, y_i))_{i=1}^{n}$ where $(x_i, y_i) \in \{(\varnothing,\varnothing), (-\square_j, +\square_h), (\varnothing, +\square_h), (-\square_h, \varnothing)\}$, $1 \leq h, j \leq k-1$ and $x_1 = y_n = \varnothing$. In the following we shall identify the sequence $(x_i, y_i)_{i=1}^{n}$ with its corresponding sequence of shapes and set

$$\varphi_1((x_i, y_i)_{i=1}^{n}) = (\tilde{x}_i, \tilde{y}_i)_{i=1}^{n-1} \quad \text{where} \quad \tilde{x}_i = y_i \ \wedge\ \tilde{y}_i = x_{i+1}\ . \tag{3.2}$$

In view of $x_1 = y_n = \varnothing$ we can conclude that φ_1 is bijective. Since the vacillating tableaux of a partition is generated by $(-\square, \varnothing)$, $(\varnothing, +\square)$, $(\varnothing, \varnothing)$, $(-\square, +\square)$, we have

$$\forall\, 1 \leq i \leq n-1; \qquad (\tilde{x}_i, \tilde{y}_i) \in \{(\varnothing, \varnothing),\ (+\square_h, \varnothing),\ (\varnothing, -\square_h),\ (+\square_h, -\square_j)\}\ , \tag{3.3}$$

where $1 \leq h, j \leq k-1$. Let φ_2 be given by

$$\varphi_2((\tilde{x}_i, \tilde{y}_i)) = \begin{cases} (\tilde{x}_i, \tilde{y}_i) & \text{for } \ (\tilde{x}_i, \tilde{y}_i) = (+\square_h, -\square_j) \\ (\tilde{y}_i, \tilde{x}_i) & \text{otherwise.} \end{cases} \tag{3.4}$$

φ_2 has by definition the property $\varphi_2((\tilde{x}_i, \tilde{y}_i)) \in \{(-\square_h, \varnothing), (\varnothing, +\square_h), (\varnothing, \varnothing), (+\square_h, -\square_j)\}$.
Claim 1. The mapping

$$\vartheta \colon \mathcal{P}_k(n) \longrightarrow \mathcal{B}_k(n-1)\ , \quad \vartheta = \beta \circ \varphi_2 \circ \varphi_1 \circ \beta^{-1}$$

is well-defined and a bijection.
For arbitrary $\pi \in \mathcal{P}_k(n)$ we set $V_{\varnothing}^{2n}(\pi)^{\dagger} = \varphi_2 \circ \varphi_1(V_{\varnothing}^{2n}(\pi))$. By construction $V_{\varnothing}^{2n}(\pi)^{\dagger}$ is given by $\varphi_2(\varphi_1(x_i, y_i)_{i=1}^{n})) = (a_i, b_i)_{i=1}^{n-1}$, where $(a_i, b_i) \in \{(-\square_h, \varnothing), (\varnothing, +\square_h), (\varnothing, \varnothing), (+\square_h, -\square_j)\}$. Its induced sequence of collections of rows of squares $(\mu^i)_{i=0}^{2(n-1)}$ has the following properties:

$$\mu^{2(n-1)} = \lambda^{2n-1} = \varnothing, \tag{3.5}$$

$$\mu^{2j+2} \setminus \mu^{2j+1},\ \mu^{2j+1} \setminus \mu^{2j} \in \{(\varnothing, \varnothing), (\varnothing, +\square), (-\square, \varnothing), (+\square, -\square)\} \tag{3.6}$$

$$\mu^{2j+1} \neq \lambda^{2j+2} \quad \Longrightarrow \quad \mu^{2j+1} \in \{\lambda^{2j+1}, \lambda^{2j+3}\}\ . \tag{3.7}$$

Eq. (3.5) is obvious and eq. (3.6) follows from eq. (3.3). By construction of $(\mu^i)_{i=0}^{2(n-1)}$, for $1 \leq j \leq n-1$, $\mu^{2j} = \lambda^{2j+1}$ holds. Suppose $\mu^{2j+1} \neq \lambda^{2j+2}$ for some $0 \leq j \leq n-2$. By definition of φ_2 only pairs containing "$\varnothing$" in at least one coordinate are transposed from which we can conclude $\mu^{2j+1} = \mu^{2j}$ or $\mu^{2j+1} = \mu^{2j+2}$, i.e.

$$\begin{tikzcd} & \lambda^{2j+1} \arrow[r] & \lambda^{2j+2} \arrow[r] & \lambda^{2j+3} \\ \mu^{2j} \arrow[ur, equal] \arrow[r, equal] & \mu^{2j+1} \arrow[ur, dash] \arrow[r] & \mu^{2j+2} \arrow[ur, equal] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & \lambda^{2j+1} \arrow[r] & \lambda^{2j+2} \arrow[r] & \lambda^{2j+3} \\ \mu^{2j} \arrow[ur, equal] \arrow[r] & \mu^{2j+1} \arrow[ur, dash] \arrow[r, equal] & \mu^{2j+2} \arrow[ur, equal] & \end{tikzcd}$$

whence eq. (3.7). In particular each collection of rows of squares μ^i is a shape, i.e. $V_{\varnothing}^{2n}(\pi)^{\dagger}$ corresponds to a braid. Eq. (3.7) immediately implies that $(\mu^i)_{1 \leq i \leq 2(n-1)}$ has at most $k-1$ rows if and only if $(\lambda^i)_{1 \leq i \leq 2n}$ does. Therefore ϑ is well-defined. Obviously ϑ is bijective and Claim 1 follows.
Claim 2. For any $\pi \in \mathcal{P}_k(n)$ holds: (i, j) is an arc of π if and only if $(i, j-1)$ is an arc in $\vartheta(\pi)$.
From the proof of Theorem 1 [6] we know that a π- and $\vartheta(\pi)$-origin at j is equivalent to the existence

of a "$+\square$" in the pair-step between the shapes λ^{2j-1} and λ^{2j} and μ^{2j-2} and μ^{2j}, respectively. We have the following alternative

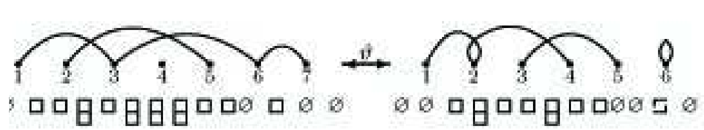
$$\begin{tikzcd}
 & \lambda^{2j-1} \arrow[r, "+\square"] & \lambda^{2j} \arrow[r, "-\square"] & \lambda^{2j+1} \\
\mu^{2j-2} \arrow[r, "+\square"] \arrow[ur, equal] & \mu^{2j-1} \arrow[r, "-\square"] \arrow[ur, dash] & \mu^{2j} \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \lambda^{2j-1} \arrow[r, "+\square"] & \lambda^{2j} \arrow[r, "\varnothing"] & \lambda^{2j+1} \\
\mu^{2j-2} \arrow[r, "\varnothing"] \arrow[ur, equal] & \mu^{2j-1} \arrow[r, "+\square"] \arrow[ur, dash] & \mu^{2j} \arrow[ur, equal] &
\end{tikzcd}$$

It is clear by diagram-chasing that π has an origin at j if and only if $\vartheta(\pi)$ does. The situation changes however for endpoints of arcs. A π- and $\vartheta(\pi)$-endpoint at j is equivalent to a "$-\square$" in the pair-step between λ^{2j-2} to λ^{2j-1} and μ^{2j-2} to μ^{2j}, respectively. Therefore we have the following two situations:

$$\begin{tikzcd}
 & \lambda^{2j-1} \arrow[r, "+\square"] & \lambda^{2j} \arrow[r, "-\square"] & \lambda^{2j+1} \\
\mu^{2j-2} \arrow[r, "+\square"] \arrow[ur, equal] & \mu^{2j-1} \arrow[r, "-\square"] \arrow[ur, dash] & \mu^{2j} \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \lambda^{2j-1} \arrow[r, "\varnothing"] & \lambda^{2j} \arrow[r, "-\square"] & \lambda^{2j+1} \\
\mu^{2j-2} \arrow[r, "-\square"] \arrow[ur, equal] & \mu^{2j-1} \arrow[r, "\varnothing"] \arrow[ur, dash] & \mu^{2j} \arrow[ur, equal] &
\end{tikzcd}$$

Again by diagram-chasing we immediately conclude that j is an endpoint in $\vartheta(\pi)$ if and only if $(j+1)$ is an endpoint in π and Claim 2 follows, completing the proof of the theorem. $\square$

As an illustration of the mapping $\vartheta\colon \mathcal{P}_k(n) \longrightarrow \mathcal{B}_k(n-1)$ we give the following example



Theorem 2 implies by restriction a bijection between k-noncrossing 2-regular partitions and braids without isolated points, denoted by $\mathcal{B}_k^{\dagger}(n)$. This is of importance since the former cannot be enumerated via the reflection principle while the latter can: braids without isolated points "just" lack the pair-step $(\varnothing, \varnothing)$ which introduces a factor e^x for the generating functions. Consequently, we can enumerate $\mathcal{B}_k^{\dagger}$ using kernel methods.

We will actually give an independent direct proof of this result. For this purpose we interpret k-noncrossing braids without isolated points as a subset of k-noncrossing partitions. For $\delta \in \mathcal{B}_k^\dagger(n)$ we identify loops with isolated points and crossing arcs (i, j) and (j, h), where $i < j < h$ by noncrossing arcs. We accordingly arrive at the bijection

$$f\colon \mathcal{B}_k^\dagger(n) \longrightarrow \{\pi \in \mathcal{P}_k(n) \mid \nexists\, (i_1, j_1), \dots, (i_k, j_k); \quad i_1 < \cdots < i_k = j_1 < \cdots < j_k\} \ . \tag{3.8}$$

Let $\mathcal{P}_{k,2}(n)$ denote the set of 2-regular, k-noncrossing partitions, i.e. the set of k-noncrossing partitions without arcs of the form $(i, i + 1)$.

Theorem 3. *Let $k \in \mathbb{N}$, $k \geq 3$. Then we have the bijection*

$$\vartheta\colon \mathcal{P}_{k,2}(n) \longrightarrow \mathcal{B}_k^\dagger(n-1) \ , \tag{3.9}$$

where ϑ is given by Theorem 2.

Proof. By construction, ϑ maps tangled-diagrams over $[n]$ into tangled diagrams over $[n-1]$. Since there exist no arcs of the form $(i, i+1)$, $\vartheta(\pi)$ is, for any $\pi \in \mathcal{P}_{k,2}(n)$ loop-free. By construction, ϑ preserves the orientation of arcs, whence $\vartheta(\pi)$ is a partition.
Claim. $\vartheta\colon \mathcal{P}_{k,2}(n) \longrightarrow \mathcal{B}_k^\dagger(n-1)$ is well-defined.
We first prove that $\vartheta(\pi)$ is k-noncrossing. Suppose there exist k mutually crossing arcs, (i_s, j_s), $s = 1, \dots, k$ in $\vartheta(\pi)$. Since $\vartheta(\pi)$ is a partition we have $i_1 < \cdots < i_k < j_1 < \cdots < j_k$. Accordingly, we obtain for the partition $\pi \in \mathcal{P}_{k,2}(n)$ the k arcs $(i_s, j_s + 1)$, $s = 1, \dots, k$ where $i_1 < \cdots < i_k < j_1 + 1 < \cdots < j_k + 1$, which is impossible since π is k-noncrossing. We next show that $\vartheta(\pi)$ is a k-noncrossing braid. If $\vartheta(\pi)$ is not a k-noncrossing braid, then according to eq. (3.8) $\vartheta(\pi)$ contains k arcs of the form $(i_1, j_1), \dots (i_k, j_k)$ such that $i_1 < \cdots < i_k = j_1 < \cdots < j_k$ holds. Then π contains the arcs $(i_1, j_1 + 1)$, $(i_k, j_k + 1)$ where $i_1 < \cdots < i_k < j_1 + 1 < \cdots < j_k + 1$, which is impossible since these arcs are a set of k mutually crossing arcs and the claim follows.
Claim. ϑ is bijective.
Clearly ϑ is injective and it remains to prove surjectivity. For any k-noncrossing braid δ there exists some 2-regular partition π such that $\vartheta(\pi) = \delta$. We have to show that π is k-noncrossing. Let $M' = \{(i_1, j_1), \dots, (i_k, j_k)\}$ be a set of k mutually crossing arcs, i.e. $i_1 < \cdots < i_k < j_1 < \cdots < j_k$. Then we have in $\vartheta(\pi)$ the arcs $(i_s, j_s - 1)$, $s = 1, \dots, k$ and $i_1 < \cdots < i_k \leq j_1 - 1 < \cdots < j_k - 1$. If $M = \{(i_1, j_1 - 1), \dots, (i_k, j_k - 1)\}$ is k-noncrossing then we can conclude $i_k = j_1 - 1$. Therefore $M = \{(i_1, j_1 - 1), \dots, (i_k, j_k - 1)\}$, where $i_k = j_1 - 1$ which is, in view of eq. (3.8) impossible in k-noncrossing braids. By transposition we have thus proved that any ϑ-preimage is necessarily a k-noncrossing partition, whence the claim and the proof of the theorem is complete. $\square$

Theorem 3 allows for lattice path enumeration of $\mathcal{P}_{k,2}(n)$. The main difficulty lies the kernel-computation [7] and at present time there exists no such formula for $k > 3$. However, for $\mathcal{B}_3^\dagger(n-1)$ we have in the following enumerative result.

Theorem 4. [1, 9] *The number of* 3*-noncrossing braids without isolated points over* $[n]$*,* $\rho_3(n)$*, is given by*

$$\begin{aligned}\rho_3(n) \quad = \quad & \sum_{s\in\mathbb{Z}} [\beta_n(1,0,s) - \beta_n(1,-1,s) - \beta_n(1,-4,s) + \beta_n(1,-3,s)\\ & -\beta_n(3,4,s) + \beta_n(3,3,s) + \beta_n(3,0,s) - \beta_n(3,1,s)\\ & +\beta_n(2,5,s) - \beta_n(2,4,s) - \beta_n(2,1,s) + \beta_n(2,2,s))]\ ,\end{aligned}$$

where $\beta_n(t,m,s) = \frac{t}{n+1}\binom{n+1}{s}\binom{n+1}{t+s}\binom{n+1}{s+m}$. *Furthermore* $\rho_3(n)$ *satisfies the recursion*

$$\alpha_1(n)\,\rho_3(n) + \alpha_2(n)\,\rho_3(n+1) + \alpha_3(n)\,\rho_3(n+2) - \alpha_4(n)\,\rho_3(n+3) = 0\ , \tag{3.10}$$

where $\alpha_1(n) = 8(n+2)(n+3)(n+1)$*,* $\alpha_2(n) = 3(n+2)(5n^2+47n+104)$*,* $\alpha_3(n) = 3(n+4)(2n+11)(n+7)$ *and* $\alpha_4(n) = (n+9)(n+8)(n+7)$ *and*

$$\rho_3(n) \sim K\ 8^n n^{-7}(1 + c_1/n + c_2/n^2 + c_3/n^3), \tag{3.11}$$

where $K = 6686.408973$*,* $c_1 = -28,\ c_2 = 455.77778$ *and* $c_3 = -5651.160494$.

The theorem has two parts: the first is the exact formula resulting from the kernel computation [1] and the second is the asymptotic formula [9]. In [1] the exact formula is computed, the authors also prove an asymptotic formula. In [9] an improved asymptotic formula is given which is based the analytic theory of singular difference equations developed by Birkhoff and Trjitzinsky [2, 3]. To keep the paper self-contained we prove Theorem 4 in the Section 4.

Remark 1. The enumeration results for $\mathcal{B}_3^\dagger(n)$ summarized in Theorem 4 imply trivially the enumeration of $\mathcal{B}_3(n)$. According to the duality between braids and partitions we have therefore obtained the enumeration of 3-noncrossing partitions.

4. Proof of Theorem 4

We have $k = 3$, i.e. walks induced by the vacillating braid-tableaux in $\mathbb{Z}^2$, starting and ending at $(1,0)$. Via the reflection principle we reduce the enumeration of these walks which remain in the first quadrant and never touch the diagonal $x = y$ to the enumeration of lattice walks in the first

quadrant starting and ending at $(1,0)$ and starting at $(1,0)$ and ending at $(0,1)$, respectively. Let $h(i,j,l)$ be the number of walks of length l that end at (i,j) and let $H(x,y;t) = \sum_{i,j,l} h(i,j,l)x^i y^j t^l$. We set $\bar{x} = x^{-1}$.

Claim 1. The series $H(x,0;t)$ and $H(0,\bar{x};t)$ satisfy

$$t^2x(x+1)H(x,0;t) = PT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2) \tag{4.1}$$

$$t^2\bar{x}(\bar{x}+1)H(0,\bar{x};t) = NT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2), \tag{4.2}$$

where the operator $PT_x(NT_x)$ extracts positive(negative) powers of x in series of $\mathbb{Q}[x,\bar{x}][[t]]$.
To prove the Claim 1 we observe that the kernel of

$$\begin{aligned} H(x,y;t) - x &= (x+y+\bar{x}+\bar{y}+x\bar{y}+y\bar{x}+y\bar{y}+x\bar{x})\, t^2 H(x,y;t) \\ &\quad -t^2\,(x\bar{y}+\bar{y})\, H(x,0;t) - t^2\,(y\bar{x}+\bar{x})\, H(0,y;t) \end{aligned}$$

is given by:

$$K_{\mathcal{B}_3}(x,y;t) = xy - t^2(x^2y + xy^2 + y + x + x^2 + y^2 + 2xy)\ . \tag{4.3}$$

$K_{\mathcal{B}_3}(x,y;t)$ is an irreducible polynomial of degree 2 over $\mathbb{Q}(y,t)$ having the two roots $Y_0 = Y_0(x,t)$ and $Y_1 = Y_1(x,t)$. Only Y_0 is a power series with positive coefficients in t^2:

$$Y_0 = \frac{1 - t^2(x+2+\bar{x}) - \sqrt{(1-t^2(x+2+\bar{x}))^2 - 4t^4x\,(1+\bar{x})^2}}{2t^2(\bar{x}+1)} \tag{4.4}$$

i.e. $Y_0(x,t) = (1+x)t^2 + (x(x+1)(\bar{x}+1)^2)t^4 + O(t^6)$. Furthermore we have $Y_0\,Y_1 = x$ and

$$x^2\,\bar{y}\,K_{\mathcal{B}_3}(\bar{x}y,y;t) = K_{\mathcal{B}_3}(x,y;t) = x^3\,K_{\mathcal{B}_3}(\bar{x}y,\bar{x};t)\ . \tag{4.5}$$

Eq. (4.5) implies $K_{\mathcal{B}_3}(\bar{x}Y_0,\bar{x};t) = K_{\mathcal{B}_3}(\bar{x}Y_0,Y_0;t) = K_{\mathcal{B}_3}(x,Y_0;t) = 0$ and we accordingly obtain

$$x^2Y_0 = t^2x(x+1)H(x,0;t) + t^2Y_0(Y_0+1)H(0,Y_0;t) \tag{4.6}$$

$$\bar{x}^2Y_0^3 = t^2\bar{x}Y_0(\bar{x}Y_0+1)H(\bar{x}Y_0,0;t) + t^2Y_0(Y_0+1)H(0,Y_0;t) \tag{4.7}$$

$$\bar{x}^3Y_0^2 = t^2\bar{x}Y_0(\bar{x}Y_0+1)H(\bar{x}Y_0,0;t) + t^2\bar{x}(\bar{x}+1)H(0,\bar{x};t)\ . \tag{4.8}$$

We next eliminate the terms $H(0,Y_0;t)$ and $H(\bar{x}Y_0,0;t)$ and arrive at

$$x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2 = t^2x(x+1)H(x,0;t) + t^2\bar{x}(\bar{x}+1)H(0,\bar{x};t)\ . \tag{4.9}$$

Since $t^2x(x+1)H(x,0;t)$ and $t^2\bar{x}H(0,\bar{x};t)$ have only positive and negative powers of x, respectively, we can conclude

$$\begin{aligned} t^2x(x+1)H(x,0;t) &= PT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2) \\ t^2\bar{x}(\bar{x}+1)H(0,\bar{x};t) &= NT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2)\ . \end{aligned}$$

Claim 2. Let CT_x denote the constant coefficient of a Laurent-series $\sum_{i\in I} a_i x^i$. Then we have

$$\rho_3(n) = [t^{2n+2}]CT_x((1-x-x^4+x^3)Y_0 + (-\bar{x}^4+\bar{x}^3+1-\bar{x})Y_0^3 + (\bar{x}^5-\bar{x}^4-\bar{x}+\bar{x}^2)Y_0^2)\ . \tag{4.10}$$

To prove Claim 2 we write $\rho_3(n) = [xt^{2n}]H(x,0;t) - [yt^{2n}]H(0,y;t)$ and interpret the terms $[xt^{2n}]H(x,0;t)$ and $[yt^{2n}]H(0,y;t)$ via eq. (4.1) and eq. (4.2):

$$\begin{aligned}
[xt^{2n}]H(x,0;t) &= [x^2t^{2n+2}]PT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2) - [xt^{2n+2}]PT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2)\\
[yt^{2n}]H(0,y;t) &= [\bar{x}t^{2n}]H(0,\bar{x};t)\\
&= [\bar{x}^2t^{2n+2}]\, NT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2) - [\bar{x}t^{2n+2}]\, NT_x(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2)\ .
\end{aligned}$$

We can combine these equations and obtain

$$\begin{aligned}
\rho_3(n) &= [t^{2n+2}]CT_x(\bar{x}^2-\bar{x}-x^2+x)(x^2Y_0 - \bar{x}^2Y_0^3 + \bar{x}^3Y_0^2)\\
&= [t^{2n+2}]CT_x((1-x-x^4+x^3)Y_0 + (-\bar{x}^4+\bar{x}^3+1-\bar{x})Y_0^3 + (\bar{x}^5-\bar{x}^4-\bar{x}+\bar{x}^2)Y_0^2)\ .
\end{aligned}$$

Claim 3. Suppose Y_0 is the solution of $K_{\mathcal{B}_3}(x,y;t)=0$ with positive coefficients in t^2 of eq. (4.4). Then we have

$$[x^m t^{2n+2}]Y_0^k = \frac{k}{n+1}\binom{n+1}{s}\binom{n+1}{k+s}\binom{n+1}{s+m}\ . \tag{4.11}$$

Since $K_{\mathcal{B}_3}(x,Y_0;t)=0$, (eq. (4.3)) we have $Y_0 = t^2(\bar{x}+1)(x+Y_0)(1+Y_0)$. Let $\mathcal{G}(t^2) = (\bar{x}+1)(x+t^2)(1+t^2)$. We derive

$$\begin{aligned}
[t^{2n+2}]Y_0^k &= \frac{k}{n+1}[t^{2(n+1-k)}](\bar{x}+1)^{n+1}(x+t^2)^{n+1}(1+t^2)^{n+1}\\
&= \frac{k}{n+1}\left(\sum_{s=0}^{n+1-k}(\bar{x}+1)^{n+1}\binom{n+1}{s}\binom{n+s}{n+1-s-k}x^{n+1-s}\right).
\end{aligned}$$

We can conclude from this

$$[x^m t^{2n+2}]Y_0^k = \frac{k}{n+1}\sum_{s=0}^{n+1}\binom{n+1}{s}\binom{n+1}{k+s}\binom{n+1}{s+m}. \tag{4.12}$$

and Claim 3 follows. In order to prove the first assertion of the theorem, we calculate the first term $[t^{2n+2}]CT_x((1-x-x^4+x^3)Y_0$ of eq. (4.10). The terms $(-\bar{x}^4+\bar{x}^3+1-\bar{x})Y_0^3$ and $(\bar{x}^5-\bar{x}^4-\bar{x}+\bar{x}^2)Y_0^2$ can be computed analogously:

$$\begin{aligned}
[t^{2n+2}]CT_x((1-x-x^4+x^3)Y_0 &= [x^0t^{2n+2}]Y_0 - [x^{-1}t^{2n+2}]Y_0 - [x^{-4}t^{2n+2}]Y_0 + [x^{-3}t^{2n+2}]Y_0\\
&= \sum_{s=0}^{n+1}(\beta_n(1,0,s) - \beta_n(1,-1,s) - \beta_n(1,-4,s) + \beta_n(1,-3,s))\ ,
\end{aligned}$$

where $\beta_n(k,m,s)=\frac{k}{n+1}\binom{n+1}{s}\binom{n+1}{k+s}\binom{n+1}{s+m}$. Using eq. (3.10) the recursion follows from Zeilberger's algorithms [8] using MAPLE.

Claim 4**.** There exist some $K>0$ and $c_1,c_2,c_3\dots$ such that

$$\rho_3(n)\sim K\ 8^n n^{-7}(1+c_1/n+c_2/n^2+c_3/n^3\cdots). \tag{4.13}$$

The theory of singular difference equations [2] guarantees the existence of 3 linearly independent formal series solutions (FSS) for eq. (3.10). We set

$$\rho_3(n)=E(n)K(n)\quad E(n)=e^{\mu_0 n\ln n+\mu_1 n}n^{\theta} \tag{4.14}$$

where $K(n)=\exp\{\alpha_1 n^{\beta+\alpha_2 n^{\beta-1/\rho+\cdots}}\}$, $\alpha_1\neq 0$, $\beta=j/\rho$, and $0\le j<\rho$. We immediately derive setting $\lambda=e^{\mu_0+\mu_1}$

$$\begin{aligned}\frac{\rho_3(n+k)}{\rho_3(n)}=n^{\mu_0 k}\lambda^k\{1+\frac{k\theta+k^2\mu_0/2}{n}+\cdots\}\\ \exp\{\alpha_1\beta k n^{\beta-1}+\alpha_2(\beta-\frac{1}{\rho})kn^{\beta-1/\rho-1+\cdots}\},\end{aligned}$$

and arrive at

$$\begin{aligned}0=1+&\frac{15}{8}\{1+\frac{\theta+\mu_0/2+\frac{27}{5}}{n}+\cdots\}\xi\{1+(\alpha_1\beta n^{\beta-1}+\alpha_2(\beta-1/\rho)n^{\beta-1/\rho-1}+\cdots)+\cdots\}\\ &+\frac{3}{4}\{1+\frac{2\theta+2\mu_0+\frac{21}{2}}{n}+\cdots\}\xi^2\{1+(2\alpha_1\beta n^{\beta-1}+2\alpha_2(\beta-1/\rho)n^{\beta-1/\rho-1}+\cdots)+\cdots\}\\ &-\frac{1}{8}\{1+\frac{3\theta+9\mu_0/2+18}{n}+\cdots\}\xi^3\{1+(3\alpha_1\beta n^{\beta-1}+3\alpha_2(\beta-1/\rho)n^{\beta-1/\rho-1}+\cdots)+\cdots\}.\end{aligned}$$

First we consider the maximum power of n, which is zero. In view of $1=\frac{1}{8}n^{3\mu_0}\lambda^3$ we obtain $\mu_0=0$. This implies $\rho=1$ since $\rho\ge 1$ and ρ should be the smallest integer s.t. $\rho\mu_0\in\mathbb{N}$. Equating the constant terms again, we obtain that λ is indeed a root of the cubic polynomial $P(X)$ $P(X)=1+\frac{15}{8}X+\frac{3}{4}X^2-\frac{1}{8}X^3$. Therefore we have $\lambda=8$ or -1. Notice that $0\le\beta<1$ implies $\beta=0$. Otherwise, equating the coefficient of $n^{\beta-1}$ implies $\alpha_1=0$, which is impossible. It remains to compute θ. For this purpose we equate the coefficient of n^{-1}, i.e. $8\frac{15}{8}(\theta+\frac{27}{5})+8^2\frac{3}{4}(\frac{21}{2}+2\theta)-8^3\frac{1}{8}(18+3\theta)=0$ from which we can conclude $\theta=-7$. Since $\rho_3(n)$ is monotone increasing $\rho_3(n)$ coincides with the only monotonously increasing FSS, given by

$$\rho_3(n)\sim K\cdot 8^n\cdot n^{-7}(1+c_1/n+c_2/n^2+c_3/n^3\cdots) \tag{4.15}$$

for some $K>0$ and constants c_1,c_2,c_3 and the proof of the Claim 4 is complete.

Equating the coefficients of $n^{-2},\ n^{-3}$ and n^{-4}, $(2268+81c_1=0,\ 1683c_1+162c_2-26712=0$ and $-32547c_1+729c_2+129654+243c_3=0)$ we obtain $c_1=-28$, $c_2=455.778$ and $c_3=-5651.160494$ and finally we compute $K=6686.408973$ numerically, completing the proof of Theorem 4.

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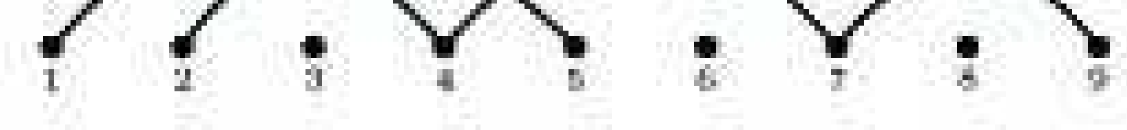
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